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Liapunov Theorem for Modular Functions

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We extend Liapunov Theorem to modular functions on complemented lattices.

INTRODUCTION

Liapunov (1940) proved that the range of a nonatomic σ-additive measure on a σ-algebra with values in a finite-dimensional vector space is convex. Later Halmos (1948) gave a simplified proof of Liapunov's result. Various versions of Liapunov's theorem appeared in the following years. More recently, Liapunov's theorem has been independently proved for nonatomic finitely additive measures on Boolean algebras with the interpolation property (see, e.g., Volkmer and Weber, 1983; Armstrong and Prikry, 1981; Candeloro and Martellotti, 1979) and for nonatomic multimeasures (see, e.g., Artstein, 1972; Avallone and Basile, 1993).

In this paper, we extend the Liapunov theorem to modular functions on complemented lattices. Precisely, we prove that, if L is a complemented lattice with the interpolation property and $\mu: L \to R^n$ is a nonatomic modular function, then, for every $a \in L$, $\mu([0, a])$ is a bounded and convex subset of R^n. A consequence is that, if X is a locally convex Hausdorff topological linear space and $\mu: L \to X$ is modular nonatomic, then the weak closure of $\mu(L)$ is convex.

In the proof, lattice uniformities (Weber, 1993a,b; Avallone and Weber, 1994) play a role similar to Frechet–Nikodym topologies in classical measure theory (Volkmer and Weber, 1983). The study of modular functions on lattices includes the study of measures on Boolean algebras, of additive functions on sectionally complemented lattices, and of linear operators on Riesz spaces.

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For a study, see, e.g., Weber (1993c–e) and Fleischer and Traynor (1980, 1982).

Preliminaries

Let L be a lattice. L is called *complemented* if L has 0 and 1 and, for every $x \in L$, there exists $x' \in L$ such that $x \vee x' = 1$ and $x \wedge x' = 0$. L is called *sectionally complemented* if L has 0 and, for every $a \in L$, $[0, a]$ is complemented. L is called *modular* if, for every x, y, z in L, $x \geq z$ implies $(x \wedge y) \vee z = x \wedge (y \vee z)$. We say that L has the *interpolation property* if, for all sequences $\{x_n\}$, $\{y_n\}$ in L such that, for every $n \in N$, $x_n \leq x_{n+1} \leq y_{n+1} \leq y_n$, there exists $x \in L$ such that, for every $n \in N$, $x_n \leq x \leq y_n$.

A *lattice uniformity* on L is a uniformity on L which makes uniformly continuous the lattice operations.

We use that, if L is sectionally complemented and $\mathcal{U}$ is a lattice uniformity on L, then a basis for $\mathcal{U}$ is the family consisting of the sets

$$\{(x, y) \in L^2: \exists a \in U_0: x \vee a = y \vee a\}$$

where U_0 is a 0-neighborhood in $\mathcal{U}$ (Avallone and Weber, 1994, Proposition 2.1).

If $(G, +)$ is a group, a function $\mu: L \to G$ is called *modular* if, for every $x, y \in L$, $\mu(x \vee y) + \mu(x \wedge y) = \mu(x) + \mu(y)$.

If G is a topological commutative group and $\mu: L \to G$ is modular, then there exists the weakest lattice uniformity $\mathcal{U}(\mu)$ that makes μ uniformly continuous and a basis for $\mathcal{U}(\mu)$ is the family consisting of the sets

$$\{(x, y) \in L^2: \mu(b) - \mu(a) \in W \text{ for every } a, b \in [x \wedge y, x \vee y], a \leq b\}$$

where W is a 0-neighborhood in G (Fleischer and Traynor, 1982; Weber, 1993d, Proposition 3.1). The $\mathcal{U}(\mu)$-topology is denoted by $\tau(\mu)$. By Weber (1993d, Proposition 2.5), $N(\mu) = \{(x, y) \in L^2: \mu \text{ is constant on } [x \wedge y, x \vee y]\}$ is a congruence, the quotient $\hat{L} = L/N(\mu)$ is a modular lattice, and, if we set $\hat{\mu}(\hat{x}) = \mu(x)$ for every $Lx \in \hat{x} \in \hat{L}$, then $\hat{\mu}: \hat{L} \to G$ is a modular function. Moreover, if L is complemented, then $\hat{L}$ is sectionally complemented.

In the following, we denote by N and R, respectively, the set of natural numbers and the set of real numbers.

1. NONATOMIC MODULAR FUNCTIONS

In this section, we prove results which will be useful in the next section to prove the Liapunov theorem.

Let L be a sectionally complemented lattice and G a topological commutative group.

Definition 1.1. Let $\mathcal{U}$ be a lattice uniformity on L. We say that $(L, \mathcal{U})$ is *chained* if, for every $a, b \in M$, with $a < b$, and for every $U \in \mathcal{U}$, there exist $x_1, \ldots, x_{n-1}$ in L such that $a = x_0 < x_1 < \cdots < x_n = b$ and $(x_{i-1}, x_i) \in U$ for $i \in \{1, \ldots, n\}$.

Definition 1.2. Let $\mathcal{U}$ be a lattice uniformity on L. $(L, \mathcal{U})$ is called *nonatomic* if, for every 0-neighborhood U_0 in $\mathcal{U}$ and for every $x \in L$, there exist $x_1, \ldots, x_n$ in L such that $x_i \wedge x_j = 0$ for $i \neq j$, $x = x_1 \vee \cdots \vee x_n$, and $x_i \in U_0$ for each $i \in \{1, \ldots, n\}$.

Proposition 1.3. Let $\mathcal{U}$ be a lattice uniformity on L. Then the following conditions are equivalent:

(1) $(L, \mathcal{U})$ is nonatomic.

(2) $(L, \mathcal{U})$ is chained.

Proof. We set $\Delta = \{(x, y) \in L^2 \colon x = y\}$.

(1) $\Rightarrow$ (2). Let $U \in \mathcal{U}$ and $x, y \in L$, with $x < y$. Moreover, let $V \in \mathcal{U}$ such that $\Delta \vee V \subseteq U$ and let c be the relative complement of x in $[0, y]$. By (1), there exist $x_1, \ldots, x_n$ in L such that $x_i \wedge x_j = 0$ for $i \neq j$, $c = x_1 \vee \cdots \vee x_n$, and $x_i \in V(0)$ for $i \in \{1, \ldots, n\}$. Set $y_0 = x$ and, for $i \in \{1, \ldots, n\}$, $y_i = x \vee \bigvee_{j=1}^{i} x_j$. Then $x = y_0 < y_1 < \cdots < y_n = x \vee c = y$ and

$$(y_{i-1}, y_i) = \left(x \vee \bigvee_{j=1}^{i-1} x_j,\, x \vee \bigvee_{j=1}^{i-1} x_j\right) \vee (0, x_i) \in \Delta \vee V \subseteq U$$

for $i \in \{1, \ldots, n\}$.

(2) $\Rightarrow$ (1). Let U_0 be a 0-neighborhood in $\mathcal{U}$ and $\bar{x} \in L$. We can suppose $U_0 = U(0)$, where $U \in \mathcal{U}$. Moreover, let $V \in \mathcal{U}$ such that $V \wedge \Delta \subseteq U$. By (2), there exist $x_1, \ldots, x_{n-1}$ in L such that $0 = x_0 < x_1 \cdots < x_n = \bar{x}$, and $(x_{i-1}, x_i) \in V$ for $i \in \{1, \ldots, n\}$. Set $y_0 = x_0$ and, for $i \in \{1, \ldots, n\}$, let y_i be the relative complement of x_{i-1} in $[0, x_i]$. Then $y_0 \vee \cdots \vee y_n = \bar{x}$, $y_i \wedge y_j = 0$ for $i \neq j$, and

$$(0, y_i) = (x_{i-1} \wedge y_i, x_i \wedge y_i) = (x_{i-1}, x_i) \wedge (y_i, y_i) \in V \wedge \Delta \subseteq U$$

from which $y_i \in U_0$ for $i \in \{1, \ldots, n\}$.

Definition 1.4. Let $\mu \colon L \to G$ be a modular function. We say that μ is nonatomic if $(L, \mathcal{U}(\mu))$ is nonatomic.

Proposition 1.5. Let $\mu \colon L \to G$ be a modular function, with $\mu(0) = 0$. Then a 0-neighborhood basis for $\mathcal{U}(\mu)$ is the family consisting of the sets $W^* = \{a \in L \colon \mu([0, a]) \subseteq W\}$, where W is a 0-neighborhood in G.

Proof. Let W be a 0-neighborhood in G and

$$W^0 = \{(x, y) \in L^2 \colon \mu(b) - \mu(a) \in W \text{ for every } a, b \in [x \wedge y, x \vee y], a \leq b\}$$

It is sufficient to prove that $W^0(0) = W^*$. Let $a \in W^0(0)$ and $x \leq a$. Let y be the relative complement of x in $[0, a]$. Then $\mu(x) = \mu(a) - \mu(y) \in W$. Hence $a \in W^*$. Conversely, let $x \in W^*$ and $a, b \in [0, x]$, with $a \leq b$. Let c be the relative complement of a in $[0, b]$. Then $\mu(b) - \mu(a) = \mu(c) \in W$. Hence $x \in W^0(0)$.

Remark. Proposition 1.5 for orthomodular lattices has been proved in Weber (1993c, Proposition 3.3).

Remark 1.6. Let μ be as in Proposition 1.5. Then, by Proposition 1.5, μ is nonatomic iff, for every 0-neighborhood W in G and for every $x \in L$, there exist $x_1, \ldots, x_n$ in L such that $x_i \wedge x_j = 0$ for $i \neq j$, $x_1 \vee \cdots \vee x_n = x$, and $\mu(y) \in W$ for every $y \leq x_i$ and $i \in \{1, \ldots, n\}$.

Remark 1.7. Let L_0 be a lattice and $\mu: L \to R$ modular, with $\mu(0) = 0$. We set, for $x \in L_0$, $\tilde{\mu}(x) = \sup\{|\mu(y)|: y \leq x\}$, $\mu^+(x) = \sup_\gamma \Sigma_{i=1}^n(\mu(x_i) - \mu(x_{i-1}))^+$, $\mu^-(x) = \sup_\gamma \Sigma_{i=1}^n(\mu(x_i) - \mu(x_{i-1}))^-$, and $|\mu|(x) = \mu^+(x) + \mu^-(x)$, where $\gamma = \{x_0, \ldots, x_n\}$ denotes a finite, totally ordered set with $x_0 = 0$ and $x_n = x$ and, for $\alpha \in R$, we set $\alpha^+ = \max\{\alpha, 0\}$, $\alpha^- = \max\{-\alpha, 0\}$. By Birkhoff (1984, X.6), it is known that, if μ is of bounded variation, then μ^+, μ^-, and $|\mu|$ are increasing modular functions and $\mu = \mu^+ - \mu^-$.

Proposition 1.8. Let $\mu: L \to R$ be modular, with $\mu(0) = 0$. Let $\tilde{\mu}$ and $|\mu|$ as in Remark 1.7. Then, for every $x \in L$, $\tilde{\mu}(x) \leq |\mu|(x) \leq 2\tilde{\mu}(x)$.

Proof. The inequality $\tilde{\mu} \leq |\mu|$ is trivial and holds in any lattice. Now let $x \in L$ and $0 = x_0 < x_1 < \cdots < x_n = x$. From Weber (1993d) (see proof of Proposition 2.8 applied to $[0, x]$), we obtain that, for every $I \subseteq \{1, \ldots, n\}$, $\Sigma_{i \in I}(\mu(x_i) - \mu(x_{i-1})) \in \mu([0, x])$. Now let $I, J \subseteq \{1, \ldots, n\}$ be such that

$$\begin{aligned}
&\sum_{i=1}^n |\mu(x_i) - \mu(x_{i-1})| \\
&\quad = \sum_{i \in I} (\mu(x_i) - \mu(x_{i-1})) - \sum_{i \in J} (\mu(x_i) - \mu(x_{i-1})) \\
&\quad \in \mu([0, x]) - \mu([0, x])
\end{aligned}$$

Then $\Sigma_{i=1}^n |\mu(x_i) - \mu(x_{i-1})| \leq 2\tilde{\mu}(x)$, from which we obtain the other inequality.

Remark. By Proposition 1.8, we obtain that μ is bounded iff μ is of bounded variation, as proved in Weber (1993d, Propositions 2.7 and 2.8).

Corollary 1.9. Let $n \in N$ and $\mu: L \to R^n$ modular, with $\mu(0) = 0$. Let $\mu = (\mu_1, \ldots, \mu_n)$, $\tilde{\mu}_i$, and $|\mu_i|$ as in Remark 1.7. Then the following conditions are equivalent:

(1) μ is nonatomic.
(2) For each $\epsilon > 0$ and $x \in L$, there exist $x_1, \ldots, x_m$ in L such that $x_i \wedge x_j = 0$ for $i \neq j$, $x_1 \vee \cdots \vee x_m = x$, and $\tilde{\mu}_i(x_j) < \epsilon$ for $i \leq n$ and $j \leq m$.
(3) For each $\epsilon > 0$ and $x \in L$, there exist $x_1, \ldots, x_m$ in L such that $x_i \wedge x_j = 0$ for $i \neq j$, $x_1 \vee \cdots \vee x_m = x$ and $|\mu_i|(x_j) < \epsilon$ for $i \leq n$ and $j \leq m$.

Proof. We use Remark 1.6 and Proposition 1.8.

Corollary 1.10. Suppose that L has 1. Let $n \in N$ and $\mu: L \to R^n$ modular and nonatomic. Then μ is bounded.

Proof. Let $\mu = (\mu_1, \ldots, \mu_n)$. We can suppose $\mu(0) = 0$, because we can replace μ by μ' defined by $\mu'(x) = \mu(x) - \mu(0)$ for $x \in L$. By Corollary 1.9, there exists $c > 0$ such that, for every $x \in L$, $|\mu_i|(x) \leq |\mu_i|(1) \leq c$ for each $i \leq n$.

2. LIAPUNOV THEOREM

In this section, let L be a lattice with the interpolation property, $n \in N$, and $\mu: L \to R^n$ a modular function, with $\mu = (\mu_1, \ldots, \mu_n)$.

The idea of the proof of Liapunov theorem (Theorem 2.3) is based on the idea of Volkmer and Weber (1983).

Lemma 2.1. Let L be sectionally complemented and modular. Suppose that, for every $i \in \{1, \ldots, n\}$, $\mu_i \geq 0$ and $\mu(0) = 0$. Then the following conditions are equivalent:

(1) For every $a \in L$, there exists $b \in L$ such that $b \leq a$ and $\mu(b) = 2^{-1}\mu(a)$.
(2) For every $a, b \in L$, there exists a continuous function α: $[0, 1] \to (L, \mathcal{U}(\mu))$ such that $\alpha(0) = a$, $\alpha(1) = b$, $\alpha(\lambda) \leq a \vee b$, and $\mu(\alpha(\lambda)) = (1 - \lambda)\mu(a) + \lambda\mu(b)$ for every $\lambda \in [0, 1]$.
(3) For every $a \in L$, $\mu([0, a])$ is convex.

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are trivial.

(1) $\Rightarrow$ (2). (i) First we prove that, for every $a \in L$, there exists a continuous and increasing function α: $[0, 1] \to (L, \mathcal{U}(\mu))$ such that $\alpha(0) = 0$, $\alpha(1) = a$, and $\mu(\alpha(\lambda)) = \lambda\mu(a)$ for every $\lambda \in [0, 1]$.

Let $a \in L\backslash\{0\}$. If $\mu(a) = 0$, set $\alpha(\lambda) = a$ for $\lambda \in [0, 1]$ and $\alpha(0) = 0$. Then α: $[0, 1] \to (L, \mathcal{U}(\mu))$ is continuous by Proposition 1.5 and satisfies (i). Therefore we can suppose $\mu(a) \neq 0$. By (1), we can inductively obtain, for every dyadic rational $k \in [0, 1]$, $a_k \in L$ such that $\mu(a_k) = k\mu(a)$ and $a_k \leq a_{k'}$ if k, k' are dyadic rational and $k \leq k'$. By the interpolation property of L, we obtain, for every $\lambda \in [0, 1]$, $a_\lambda \in L$ such that $\mu(a_\lambda) = \lambda\mu(a)$ and $a_\lambda \leq a_{\lambda'}$ if $\lambda \leq \lambda'$. Set $\alpha(\lambda) = a_\lambda$ for every $\lambda \in [0, 1]$. We prove that α: $[0, 1] \to (L, \mathcal{U}(\mu))$ is continuous. We use Proposition 1.5 and Proposition 2.1 of Avallone and Weber (1994). Let $\lambda_k, \lambda \in [0, 1]$ such that $\lambda_k \to \lambda$. Let $\epsilon > 0$ and $\nu \in N$ such that, for every $k > \nu$, $|\lambda_k - \lambda| < \epsilon/\|\mu(a)\|$. Let $k > \nu$. First suppose $\lambda_k \leq \lambda$. Then, denoting by b_k the relative complement of a_{λ_k} in $[0, a_\lambda]$, we have $a_{\lambda_k} \vee b_k = a_\lambda \vee b_k$ and $\mu(b_k) = \mu(a_\lambda) - \mu(a_{\lambda_k}) = (\lambda - \lambda_k)\mu(a)$, from which $\mu_i(y) < \epsilon$ for every $i \leq n$ and $y \leq b_k$. In similar way, if $\lambda_k > \lambda$, we obtain c_k such that $a_\lambda \vee c_k = a_{\lambda_k} \vee c_k$ and $\mu_i(y) < \epsilon$ for every $i \leq n$ and $y \leq c_k$.

(ii) Now let a, b in L and let c, d be the relative complements of $a \wedge b$, respectively, in $[0, a]$ and in $[0, b]$. By (i), for every $i \in \{1, 2\}$ there exist continuous and increasing functions α_i: $[0, 1] \to (L, \mathcal{U}(\mu))$ such that $\alpha_1(0) = \alpha_2(0) = 0$, $\alpha_1(1) = c$, $\alpha_2(1) = d$, $\mu(\alpha_1(\lambda)) = \lambda\mu(c)$, and $\mu(\alpha_2(\lambda)) = \lambda\mu(d)$. Set, for $\lambda \in [0, 1]$, $\alpha(\lambda) = \alpha_1(1 - \lambda) \vee \alpha_2(\lambda) \vee (a \wedge b)$. Then, using that $\alpha_1(1 - \lambda) \wedge \alpha_2(\lambda) \leq c \wedge d = 0$ and, by the modularity of L,

$$\begin{aligned}(\alpha_1(1 - \lambda) \vee \alpha_2(\lambda)) \wedge (a \wedge b) &\leq (c \vee d) \wedge (a \wedge b)\\ &= ((a \wedge d) \vee c) \wedge b = c \wedge b = 0\end{aligned}$$

we see that α satisfies (2).

Lemma 2.2. Let L be sectionally complemented and modular. Suppose $\mu_i \geq 0$ for every $i \leq n$ and $\mu(0) = 0$. Let ν: $L \to [0, +\infty]$ be a modular function with $\nu(0) = 0$ such that ν is continuous with respect to $\tau(\mu)$. Then the following conditions are equivalent:

(1) For every $a \in L$, $\mu([0, a])$ is convex.
(2) For every $a \in L$, $(\mu, \nu)([0, a])$ is convex.

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2). Let $a \in L$ and set $\mu' = (\mu, \nu)$. By Lemma 2.1, there exists $b \leq a$ such that $\mu(b) = 2^{-1}\mu(a)$. Let c be the relative complement of b in $[0, a]$. Again by Lemma 2.1, there exists a continuous α: $[0, 1] \to (L, \mathcal{U}(\mu))$ such that $\alpha(0) = b$, $\alpha(1) = c$, $\alpha(\lambda) \leq a$, and $\mu(\alpha(\lambda)) = (1 - \lambda)\mu(b) + \lambda\mu(c)$ for $\lambda \in [0, 1]$. Then, for $\lambda \in [0, 1]$, $\mu(\alpha(\lambda)) = 2^{-1}\mu(a)$. By the continuity of $\nu \circ \alpha$, we obtain $\lambda_0 \in [0, 1]$ such that $\nu(\alpha(\lambda_0)) = 2^{-1}\nu(a)$. Therefore, set $d = \alpha(\lambda_0)$, $d \leq a$, and $\mu'(d) = 2^{-1}\mu'(a)$. By Lemma 2.1, $\mu'([0, a])$ is convex.

Theorem 2.3. Let L be complemented and μ nonatomic. Then, for every $a \in L$, $\mu([0, a])$ is a bounded and convex subset of R^n.

Proof. As in Corollary 1.10, we can suppose $\mu(0) = 0$. Moreover, we can replace L by $\hat{L}$ and μ by $\hat{\mu}$ (it is easy to show that $\hat{L}$ has the interpolation property and $\hat{\mu}$ is nonatomic). Therefore we can suppose L sectionally complemented and modular.

Let $a \in L$. By Corollary 1.10, $\mu([0, a])$ is bounded. First suppose $\mu_i \geq 0$ for every $i \leq n$. For $n = 1$, the result, for Proposition 1.3, follows from Weber (1993d, Theorem 5.11). Then we use induction. Let $\mu = (\mu_1, \ldots, \mu_n)$ be nonatomic, with $\mu_i \geq 0$. Set $\mu' = (\mu_1 + \mu_n, \mu_2, \ldots, \mu_{n-1})$. Then $\mu': L \to R^{n-1}$ is by Corollary 1.9 nonatomic. By the induction assumption, $\mu'([0, a])$ is convex. Since μ_n is continuous with respect to $\tau(\mu')$, by Lemma 2.2, $(\mu', \mu_n)([0, a])$ is convex. If we set $T(t_1, \ldots, t_n) = (t_1 - t_n, t_2, \ldots, t_n)$, then $T: R^n \to R^n$ is linear and $T(\mu', \mu_n)([0, a]) = \mu([0, a])$. Therefore $\mu([0, a])$ is convex.

Now we remove the assumption $\mu_i \geq 0$ for $i \leq n$. By Remark 1.7 and Proposition 1.8, for each $i \leq n$ there exist modular functions $\lambda_i, \nu_i: L \to [0, +\infty[$, with $\lambda_i, \nu_i \leq |\mu_i|$ and $\mu_i = \lambda_i - \nu_i$. Set $\mu' = (\lambda_1, \ldots, \lambda_n, \nu_1, \ldots, \nu_n)$. By Corollary 1.9, $\mu': L \to R^{2n}$ is a nonatomic modular function. Then $\mu'([0, a])$ is convex. Set $T: (z_1, z_2) \in R^{2n} \to z_1 - z_2 \in R^n$. Hence T is linear and $T(\mu'([0, a])) = \mu([0, a])$. Therefore $\mu([0, a])$ is convex.

Corollary 2.4. Let L be complemented, X a locally convex Hausdorff topological linear space, and $\mu': L \to X$ a nonatomic modular function. Then, for every $a \in L$, the weak closure of $\mu'([0, a])$ is convex.

Proof. For $A \subseteq X$, we denote by conv A the convex hull of A and by $\overline{A}^w$ the weak closure of A. Moreover, let X' be the topological dual of X.

Let $a \in L$. We prove that conv $\mu'([0, a]) \subseteq \overline{\mu'([0, a])}^w$. Let $y \in$ conv $\mu'([0, a])$ and $f_1, \ldots, f_n \in X'$. Set $\nu = (f_1 \circ \mu', \ldots, f_n \circ \mu')$. Then $\nu: L \to R^n$ is a nonatomic modular function, because $\tau(\nu) \leq \tau(\mu)$ [we use that $\tau(\nu)$ is the weakest locally convex topology on L which makes ν continuous; see Weber (1993d, Proposition 3.2)]. By Theorem 2.3, $\nu([0, a])$ is convex. Then $(f_1(y), \ldots, f_n(y)) \in$ conv $\nu([0, a]) = \nu([0, a])$. Therefore there exists $x \leq a$ such that $f_i(y) = f_i(\mu'(x))$ for each $i \leq n$. Hence $y \in \overline{\mu'([0, a])}^w$.

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